MISGUIDED RIEMANN HYPOTHESIS

The Riemann hypothesis is regarded as one of the most important unsolved mathematical problems; however, it is false because it is ill-defined.

1. Traditional Zeta-Eta Identity

Consider the traditional Riemann zeta function.

$$\mathbb{C}:\ \zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots. \tag{1.1}$$

The Riemann zeta function converges (or is defined) for Re(s) > 1.

Multiply the Riemann zeta function by $2/2^s$.

$$\mathbb{C}:\ \frac{2}{2^s}\cdot\zeta(s) = \frac{2}{2^s}\cdot\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots\right) = \frac{2}{2^s\cdot 1^s} + \frac{2}{2^s\cdot 2^s} + \frac{2}{2^s\cdot 3^s} + \cdots = \frac{2}{2^s} + \frac{2}{4^s} + \frac{2}{6^s} + \cdots. \tag{1.2}$$

Subtract the new series from the Riemann zeta function.

$$\begin{aligned}\mathbb{C}:\ \zeta(s) - \frac{2}{2^s}\cdot\zeta(s) &= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \cdots - \left(\frac{2}{2^s} + \frac{2}{4^s} + \frac{2}{6^s} + \cdots\right)\\ &= \frac{1}{1^s} + \left(\frac{1}{2^s} - \frac{2}{2^s}\right) + \frac{1}{3^s} + \left(\frac{1}{4^s} - \frac{2}{4^s}\right) + \frac{1}{5^s} + \left(\frac{1}{6^s} - \frac{2}{6^s}\right) + \cdots\\ &= \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} + \cdots.\end{aligned} \tag{1.3}$$

The alternating infinite series is defined as the Eta function.

$$\mathbb{C}:\ \eta(s) = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} + \cdots. \tag{1.4}$$

Therefore,

$$\mathbb{C}:\ \left(1 - \frac{2}{2^s}\right)\cdot\zeta(s) = \eta(s). \tag{1.5}$$

$$\mathbb{C}:\ (1 - 2^{1-s})\cdot\zeta(s) = \eta(s). \tag{1.6}$$

Alternatively,

$$\mathbb{C}:\ \zeta(s) = \frac{\eta(s)}{1 - 2^{1-s}}. \tag{1.7}$$

Now, the Riemann zeta function also converges (or is defined) for 0 < Re(s) < 1.

This Zeta-Eta identity is the core element of the analytic continuation and Riemann hypothesis. Despite its status in modern mathematics, this Zeta-Eta identity is a fundamentally erroneous mathematical statement. As a result, the Riemann hypothesis is false as it is ill-defined.

2. Zeta-Eta Identity in the Polarized Number System

To understand the flaws, consider the Riemann zeta function in the polarized number system.

$$\mathbb{P}\text{:}\ \ \zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{\infty^s}. \tag{2.1}$$

Contrary to the traditional expression, the last term of an infinite series is listed in the polarized number system. Even though infinity cannot be expressed as a finite number, it must be treated as one in mathematical statements such as infinite series. Therefore, an infinite series must have an explicit beginning and end.

Now, consider a slightly modified expression of the Riemann zeta function.

$$\mathbb{P}\text{:}\ \ \zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{\left(\frac{\infty}{2}\right)^s} + \frac{1}{\left(\frac{\infty}{2}+1\right)^s} + \frac{1}{\left(\frac{\infty}{2}+2\right)^s} + \cdots + \frac{1}{\infty^s}. \tag{2.2}$$

Divide the infinite series into halves.

$$\mathbb{P}\text{:}\ \ \zeta_1^{\frac{\infty}{2}}(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{\left(\frac{\infty}{2}\right)^s}. \tag{2.3}$$

$$\mathbb{P}\text{:}\ \ \zeta_{\frac{\infty}{2}+1}^{\infty}(s) = \frac{1}{\left(\frac{\infty}{2}+1\right)^s} + \frac{1}{\left(\frac{\infty}{2}+2\right)^s} + \frac{1}{\left(\frac{\infty}{2}+3\right)^s} + \cdots + \frac{1}{\infty^s}. \tag{2.4}$$

As a result, the Riemann zeta function can be rewritten in detail as follows:

$$\mathbb{P}\text{:}\ \ \zeta_1^{\infty}(s) = \underbrace{\underbrace{\left[\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{\left(\frac{\infty}{2}\right)^s}\right]}_{\frac{\infty}{2}\ terms} + \underbrace{\left[\frac{1}{\left(\frac{\infty}{2}+1\right)^s} + \frac{1}{\left(\frac{\infty}{2}+2\right)^s} + \cdots + \frac{1}{\infty^s}\right]}_{\frac{\infty}{2}\ terms}}_{\infty\ terms}. \tag{2.5}$$

Multiply the Riemann zeta function by $2/2^s$.

$$\mathbb{P}\text{:}\ \ \frac{2}{2^s}\cdot\zeta_1^{\infty}(s) =$$

$$= \frac{2}{2^s}\cdot\left[\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{\left(\frac{\infty}{2}\right)^s}\right] + \frac{2}{2^s}\cdot\left[\frac{1}{\left(\frac{\infty}{2}+1\right)^s} + \frac{1}{\left(\frac{\infty}{2}+2\right)^s} + \cdots + \frac{1}{\infty^s}\right] \tag{2.6}$$

$$= \left[\frac{2}{2^s} + \frac{2}{4^s} + \frac{2}{6^s} + \cdots + \frac{2}{\infty^s}\right] + \left[\frac{2}{(\infty+2)^s} + \frac{2}{(\infty+4)^s} + \frac{2}{(\infty+6)^s} + \cdots + \frac{2}{(2\infty)^s}\right].$$

Subtract the new series from the Riemann zeta function.

$$\mathbb{P}\text{:}\ \ \zeta_1^{\infty}(s) - \frac{2}{2^s}\cdot\zeta_1^{\infty}(s)$$

$$= \underbrace{\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \cdots + \frac{1}{\left(\frac{\infty}{2}\right)^s} + \frac{1}{\left(\frac{\infty}{2}+1\right)^s} + \cdots + \frac{1}{\infty^s}}_{\infty\ terms} \tag{2.7}$$

$$- \underbrace{\left[\frac{2}{2^s} + \frac{2}{4^s} + \frac{2}{6^s} + \cdots + \frac{2}{\infty^s}\right]}_{\frac{\infty}{2}\ terms} - \underbrace{\left[\frac{2}{(\infty+2)^s} + \frac{2}{(\infty+4)^s} + \frac{2}{(\infty+6)^s} + \cdots + \frac{2}{(2\infty)^s}\right]}_{\frac{\infty}{2}\ terms}.$$

To understand how the subtraction plays out, pair the terms with even bases in the denominator.

$$\mathbb{P}:\ \zeta_1^{\infty}(s) - \frac{2}{2^s}\cdot\zeta_1^{\infty}(s)$$
$$= \left[\frac{1}{1^s} + \left(\frac{1}{2^s} - \frac{2}{2^s}\right) + \frac{1}{3^s} + \left(\frac{1}{4^s} - \frac{2}{4^s}\right) + \frac{1}{5^s} + \left(\frac{1}{6^s} - \frac{2}{6^s}\right) + \cdots + \left(\frac{1}{\infty^s} - \frac{2}{\infty^s}\right)\right] \tag{2.8}$$
$$- \left[\frac{2}{(\infty+2)^s} + \frac{2}{(\infty+4)^s} + \frac{2}{(\infty+6)^s} + \cdots + \frac{2}{(2\infty)^s}\right].$$

All terms of the Riemann zeta function with even bases in the denominator can be paired with the product of the first half of the Riemann zeta function and $2/2^s$. However, the product of the second half of the Riemann zeta function and $2/2^s$ yields terms that do not have counterparts in the original Riemann zeta function.

Therefore, the subtraction yields the following:

$$\mathbb{P}:\ \left(1 - \frac{2}{2^s}\right)\cdot\zeta_1^{\infty}(s) = \underbrace{\left[\frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} + \cdots - \frac{1}{\infty^s}\right]}_{\infty\ terms} - \frac{2}{2^s}\cdot\underbrace{\zeta_{\frac{\infty}{2}+1}^{\infty}(s)}_{\frac{\infty}{2}\ terms}. \tag{2.9}$$

Suppose the alternating infinite series is defined as the Eta function.

$$\mathbb{C}:\ \eta(s) = \underbrace{\frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} + \cdots - \frac{1}{\infty^s}}_{\infty\ terms}. \tag{2.10}$$

Then,

$$\mathbb{P}:\ (1 - 2^{1-s})\cdot\zeta(s) = \eta(s) - \frac{2}{2^s}\cdot\zeta_{\frac{\infty}{2}+1}^{\infty}(s). \tag{2.11}$$

The traditional derivation focuses only on the beginnings of infinite series and fails to account for their tail ends. As a result, it is incorrectly assumed that the subtraction yields only the alternating terms.

$$\mathbb{C}:\ (1 - 2^{1-s})\cdot\zeta(s) = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} + \cdots. \tag{2.12}$$

Again, the traditional derivation is gravely flawed. Equation (2.12) is a false mathematical statement, and it should not be used in any scenarios, including but not limited to the analytic continuation of the Riemann zeta function.

On the contrary, the Zeta-Eta identity within the framework of the polarized number system in Equation (2.11) is a true mathematical statement. It does not lead to results that contradict common sense or established mathematical principles. Therefore,

$$\mathbb{P}:\ \zeta(s) = \frac{1}{1 - 2^{1-s}}\cdot\eta(s) - \frac{2^{1-s}}{1 - 2^{1-s}}\cdot\zeta_{\frac{\infty}{2}+1}^{\infty}(s). \tag{2.13}$$

Nevertheless, this identity is meaningless because the Riemann zeta function, whether complete or partial, is essentially on both sides of the equality sign.

3. Conclusion

The Riemann hypothesis is false because it is ill-defined.